Discrete spectrum for n-cell potentials.

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Abstract

We study the scattering problem, the Sturm-Liouville problem and the spectral problem with periodic or skew-periodic boundary conditions for the one-dimensional Schrödinger equation with an n-cell (finite periodic) potential. We give explicit upper and lower bounds for the distribution functions of discrete spectrum for these problems. For the scattering problem we give, besides, explicit upper and lower bounds for the distribution function of discrete spectrum for the case of potential consisting of n not necessarily identical cells. For the scattering problem some results about transmission resonances are obtained.

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0. Introduction.

We consider the one-dimensional Schrödinger equation

$$-\Psi'' + q_n(x)\Psi = E\Psi \tag{0.1}$$

with an n-cell (finite-periodic) potential $q_n(x)$, i.e. $q_n(x) = \chi_n(x)q(x)$, where $q(x)$ is a real-valued integrable periodic potential with period a, $\chi_n(x)$ denotes the characteristic function of the interval $[0, na]$, $n \in \mathbb{N}$. We study

1. the scattering problem on the whole line
2. the Sturm-Liouville problem on the interval $[0, na]$, i.e. the spectral problem on $[0, na]$ with the boundary conditions

$$\Psi(0)\cos\alpha - \Psi'(0)\sin\alpha = 0 \tag{0.2}$$

$$\Psi(na)\cos\beta - \Psi'(na)\sin\beta = 0, \qquad \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R},$$

   and

3. the spectral problem on $[0, na]$ with periodic ($0.3a$) or skew-periodic ($0.3b$) boundary conditions

$$\Psi(0) = \Psi(na), \qquad \Psi'(0) = \Psi'(na), \tag{0.3a}$$

$$\Psi(0) = -\Psi(na), \qquad \Psi'(0) = -\Psi'(na), \tag{0.3b}$$

We discuss relations between spectral data for these problems and spectral data for the one-dimensional Schrödinger equation on the whole line with the related periodic potential q.

In the present paper we obtain, in particular, the following estimates:

- if $F_{sc}^{(n)}(\Omega)$ is the distribution function of discrete spectrum for the scattering problem for (0.1) on the whole line (the number of eigenvalues in $\Omega \subset]-\infty, 0]$ for this problem), then

$$\left|F_{sc}^{(n)}(]-\infty, E[) - [\pi^{-1}nap(E)]\right| \le 1 \qquad \text{for } E \le 0, \tag{0.4}$$

- if $F^{(n)}(\Omega)$ is the distribution function of discrete spectrum for (0.1) on $[0, na]$ with (0.2) (the number of eigenvalues in $\Omega \subset \mathbb{R}$ for this problem), then

$$\left|F^{(n)}(]-\infty, E])-[\pi^{-1}nap(E)]\right| \leq 1 \qquad \text{for } E \in \mathbb{R},\ 0 \leq \alpha \leq \beta \leq \pi,\ \beta \neq 0,\ \alpha \neq \pi, \tag{0.5a}$$

$$\left|F^{(n)}(]-\infty, E[)-[\pi^{-1}nap(E)]-1\right| \leq 1 \qquad \text{for } E \in \mathbb{R},\ \ 0 < \beta < \alpha < \pi, \tag{0.5b}$$

- if $F^{(n)}(\Omega)$ is the distribution function of discrete spectrum for (0.1) on $[0, na]$ with $(0.3a)$ or $(0.3b)$ (the sum of multiciplities of eifenvalues in $\Omega \subset \mathbb{R}$ for this problem), then

$$[\pi^{-1}nap(E)] \leq F^{(n)}(]-\infty, E[) \leq [\pi^{-1}nap(E)]+1 \qquad \text{for } E \in \mathbb{R}, \tag{0.6}$$

  where $p(E)$ is the real part of the global quasimomentum for the related periodic potential $q(x)$, $[r]$ is the integer part of $r \geq 0$.

To obtain $(0.4), (0.5)$ we use, in particular, the technique presented in Chapter 8 of [CL] and some arguments of [JM]. The estimate (0.6) follows, actually, from well-known results presented in Chapter 21 of [T].

The estimates (0.4) – (0.6) and additional estimates for the distribution functions of discrete spectrum are given in Theorems 1, 2, Corollaries 1, 2, and by the formulas $(2.22), (2.23), (2.29)$ – (2.32) in Section 2 of the present paper.

As a corollary of $(0.5), (0.6)$ one can obtain the following formula of [Sh]:

$$\lim_{n\to\infty}(na)^{-1}F^{(n)}(]-\infty, E]) = \pi^{-1}p(E), \quad E \in \mathbb{R}, \tag{0.7}$$

where $F^{(n)}(\Omega)$ is the distribution function for (0.1) on $[0, na]$ with (0.2) or $(0.3a)$ or $(0.3b)$, $p(E)$ is the real part of the global quasimomentum for the related periodic potential. The formula (0.7) for the case of smooth potential is a particular case of results of [Sh] about density of states of multidimensional selfadjoint elliptic operators with almost periodic coefficients. The formula (0.7) (for the case of continuous potential) follows from result of [JM] about density of states for the one-dimensional Schrödinger equation with almost periodic potential. Note that the methods of [Sh] and [JM] are very different.

Remark. Less precise estimates instead of (0.4) and (0.5) follow directly from (0.6) and well-known results (see Theorems 1.1, 2.1, 3.1 of Chapter

8 of [CL] and, for example, §1 of Chapter 1 of [NMPZ]) about zeros of eigenfunctions of the one-dimensional Schrödinger operator. Probably, one can generalize such an approach to the multidimensional case. Concerning the distribution function of discrete spectrum for the multidimensional Schrödinger operator with a finite periodic potential with periodic boundary conditions see the proof of Theorem XIII.101 of [RS]. Concerning results about zeros of eigenfunctions of multidimensional Schrödinger operator see [Ku] and subsequent references given there and also §6 of Chapter VI of [CH].

The transmission resonances for the scattering problem for (0.1) on the whole line are also considered in the present paper. An energy E is a transmission resonance iff $E \in \mathbb{R}_+$ and the reflection coefficients are equal to zero at this energy. The main features of the transmission resonances for an n-cell scatterer were discussed in [SWM], [RRT]. In the present paper (Proposition 1, the formula (2.25)) we give the following additional results about the transmission resonances: if $E \in \mathbb{R}_+$ is a double eigenvalue for (0.1) on $[0, na]$ with (0.3a) or (0.3b), then E is a transmission resonances, and all n-dependent transmission resonances have this origin; there are no transmission resonances in the forbidden energy set for the the related periodic potential; if $q(x) \not\equiv 0$, then

$$(na)^{-1}\Phi_{sc}^{(n)}(]0, E]) - \pi^{-1}(p(E) - p(0)) = O(n^{-1})$$

as $n \to \infty$, where $\Phi_{sc}^{(n)}(\Omega)$ is the number of transmission resonances in $\Omega \subset \mathbb{R}_+$ for an n-cell scatterer, $p(E)$ is the real part of the global quasimomentum for related periodic potential $q(x)$.

We consider also the one-dimensional Schrödinger equation

$$-\psi'' + q(x)\psi = E\psi, \quad x \in \mathbb{R}, \tag{0.8}$$

with a potential consisting of n not necessarily identical cells. More precisely, we suppose that : $\mathbb{R} = \cup_{j=1}^{n} I_j$, where $I_1 =]-\infty, x_1]$, $I_j = [x_{j-1}, x_j]$ for $1 < j < n$, $I_n = [x_{n-1}, +\infty[$, $-\infty < x_{j-1} < x_j < +\infty$ for $1 < j < n$; $q(x) = \sum_{j=1}^{n} q_j(x)$, where $q_j \in L^1(\mathbb{R})$, $q_j = \bar{q}_j$, $supp\, q_j \subseteq I_j$ for $1 \le j \le n$ and, in addition, $(1 + |x|)q_1(x)$ and $(1 + |x|)q_n(x)$ are also integrable on $\mathbb{R}$.

In the present paper (Theorem 3) we obtain, in particular, the following estimate

$$|F(] - \infty, E[) - \sum_{j=1}^{n} F_j(] - \infty, E[)| \le n - 1 \quad \text{for} \quad E \le 0, \tag{0.9}$$

where $F(]-\infty, E[)$, $(F_j(]-\infty, E[)$, resp.) denotes the distribution function of discrete spectrum for the scattering problem for (0.8) (for the one-dimensional Schrödinger equation with the potential q_j, resp.) on the whole line.

In addition, for $E = 0$ we have the estimate (2.36) obtained earlier in [AKM2] as a development of results of [K] and [SV].

Additional indications conserning preceeding works are given in Section 2 of the present paper. In connection with results discussed in the present paper it is useful to see also the review given in §17 of [RSS] and [KS] and the results given in [ZV].

1. Definitions, notations, assumptions and some known facts.

We consider the one-dimensional Schrödinger equation

$$-\frac{d^2}{dx^2}\Psi + q_n(x)\Psi = E\Psi, \quad x \in \mathbb{R}, \tag{1.1}$$

where $q_n(x)$ is an n-cell potential, i.e.

$$q_n(x) = \sum_{j=0}^{n-1} q_1(x - ja), \qquad a \in \mathbb{R}_+, \tag{1.2}$$

$$q_1 \in L^1(\mathbb{R}), \qquad q_1 = \bar{q}_1, \qquad \operatorname{supp} q_1 \in [0, a]. \tag{1.3}$$

First, we consider the scattering problem for the equation (1.1) on the whole line: we consider wave functions describing scattering with incident waves for positive energies and bound states for negative energies. We recall some definitions and facts of the scattering theory for the Schrödinger equation

$$-\Psi'' + v(x)\Psi = E\Psi \tag{1.4}$$

where

$$v \in L^1(\mathbb{R}), \quad v = \bar{v}, \qquad \int_{\mathbb{R}} (1 + |x|)|v(x)|dx < \infty \tag{1.5}$$

(see, for example, [F]). Let an incident wave be described by $e^{ikx}, k \in \mathbb{R}, k^2 = E > 0$. Then the scattering is described by the wave function $\Psi^+(x,k)$ defined as a solution of (1.4) such that

$$\Psi^+(x,k) = e^{ikx} - \frac{\pi i}{|k|} e^{i|k||x|} f(k, |k|\frac{x}{|x|}) + o(1) \quad \text{as } x \to \infty \tag{1.6}$$

for some $f(k,l), l \in \mathbb{R}, l^2 = k^2$, which is the scattering amplitude. The following formulas connect the scattering amplitude f and the scatering matrix $S(k) = (s_{ij}(k)), k \in \mathbb{R}_+$:

$$\begin{aligned} &s_{11}(k) = 1 - \pi i k^{-1} f(-k,-k), \quad s_{12}(k) = -\pi i k^{-1} f(-k,k), \\ &s_{21}(k) = -\pi i k^{-1} f(k,-k), \quad s_{22}(k) = 1 - \pi i k^{-1} f(k,k). \end{aligned} \tag{1.7}$$

The bound states energies E_j are defined as the discrete spectrum and the bound states $\Psi_j(x)$ are defined as related eigenfunctions of the spectral problem (1.4) in $L^2(\mathbb{R})$. We recall that, under assumption (1.5),

$$S(k), \quad k \in \mathbb{R}_+, \quad \text{is unitary and} \quad s_{11}(k) = s_{22}(k), \tag{1.8}$$

each eigenvalue E_j is negative and simple and the total number m of these eigenvalues is finite

$$E_1 < E_2 < \ldots < E_m < 0, \quad m < \infty. \tag{1.9}$$

For $v = q_n$ we will write $\Psi^+, f, S, s_{ij}, E_j, \Psi_j$ as $\Psi_n^+, f_n, S_n, s_{ij}^{(n)}, E_j^{(n)}, \Psi_j^{(n)}$.

Second, we consider the spectral problem (1.1) on the interval $[0, na]$ with the boundary conditions

$$\Psi(0)\cos\alpha - \Psi'(0)\sin\alpha = 0$$

$$\Psi(na)\cos\beta - \Psi'(na)\sin\beta = 0, \quad \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}. \tag{1.10}$$

Without loss of generality we may assume that

$$0 \le \alpha < \pi, \quad 0 < \beta \le \pi. \tag{1.11}$$

Third, we consider the spectral problem (1.1) on the interval $[0, na]$ with the boundary conditions

$$\Psi(0) = \Psi(na), \quad \Psi'(0) = \Psi'(na) \tag{1.12a}$$

or with the boundary conditions

$$\Psi(0) = -\Psi(na), \quad \Psi'(0) = -\Psi'(na). \tag{1.12b}$$

On the other hand, we consider the one-dimensional Schrödinger equation

$$-\frac{d^2}{dx^2}\Psi + q(x)\Psi = E\Psi, \quad x \in \mathbb{R}, \tag{1.13}$$

where q is the following periodic potential

$$q(x) = \sum_{j=-\infty}^{\infty} q_1(x - ja). \tag{1.14}$$

We recall some definitions and facts of spectral theory for the equation (1.13) on the whole line (see, for example, §17 of [RSS] and Chapter II of [NMPZ]).

The monodromy operator $M(E)$ is defined as the translation operator by the period a in the two-dimensional space of solutions of (1.13) at fixed E. If a basis in this space is fixed, then one can consider $M(E)$ as a 2×2 matrix. For all E $det M(E) = 1$. The eigenvalues of $M(E)$ are of the form

$$\lambda_1 = 1/\lambda_2 = \left(Tr\ M(E) + \sqrt{((Tr\ M(E))^2 - 4}\right)/2 = e^{i\varphi(E)}, \tag{1.15}$$

where

$$2\cos\varphi(E) = Tr\ M(E). \tag{1.16}$$

The Bloch solutions are defined as eigenvectors of $M(E)$.

The allowed and forbidden Bloch zones are defined by the formulas

$$\bigcup_{j\in\ J} \Lambda_j^a = \Lambda^a = \{E \in \mathbb{R} |\, |Tr\ M(E)| \leq 2\} \qquad \text{allowed zones} \tag{1.17}$$

$$\bigcup_{j\in\ J} \Lambda_j^f = \Lambda^f = \{E \in \mathbb{R} |\, |Tr\ M(E)| > 2\} \qquad \text{forbidden zones} \tag{1.18}$$

where either $J = \mathbb{N}$ or $J = \{1, \ldots, m\}$, $m \in \mathbb{N}$; Λ_j^a, Λ^f are connected intervals (closed for the case (1.17) and open for the case (1.18)) such that

$$\sup_{E\in\Lambda_j^a} E < \inf_{E\in\Lambda_{j+1}^a} E, \quad \sup_{E\in\Lambda_j^f} E < \inf_{E\in\Lambda_{j+1}^f} E, \quad \text{for} \quad j,\, j+1 \in J, \tag{1.19}$$

in addition, $\Lambda_1^f =]-\infty, \lambda_0[$.

The real part of the global quasimomentum $p(E)$ is defined as a real-valued continuous nondecreasing function such that: $p(E)$ is constant in each forbidden zone Λ_j^f, $p(E) = 0$ for $E \in \Lambda_1^f$, the phase $\varphi(E) = ap(E)$ is a solution of (1.16) for each allowed zone Λ_j^a. Note that

$$\pi^{-1} ap(E) = l_j \in \mathbb{N} \cup 0 \ \text{ for } E \in \bar{\Lambda}_j^f \tag{1.20}$$

(the closure of Λ_j^f) for each $j \in J$, $l_1 = 0$, $l_j < l_{j+1}$ for $j, j+1 \in J$.

2. The main new results and the preceeding results.

In the present paper we discuss relations between spectral data for (1.1) for the first, the second or the third case described above and spectral data for (1.13).

To start with, we discuss some results of [SWM], [RRT], [R], [SV]. In [SWM] the following formulas are given, in particular:

$$\frac{R_n}{T_n} = \frac{\sin n\varphi}{\sin \varphi}\ \frac{R_1}{T_1} \tag{2.1}$$

$$\begin{aligned} \tfrac{1}{T_n} &= \frac{1}{\sin\varphi}\left(\tfrac{1}{T_1}\ \sin n\varphi - \sin{(n-1)\varphi}\right), \\ R_n &= R_1 \left(1 - \frac{\sin{(n-1)\varphi}}{\sin\varphi} T_1\right)^{-1}, \end{aligned} \tag{2.2}$$

$$M(E) = \begin{pmatrix} \dfrac{1}{T_1} & -\dfrac{\bar{R}_1}{\bar{T}_1} \\ & \\ -\dfrac{R_1}{T_1} & \dfrac{1}{\bar{T}_1} \end{pmatrix} \tag{2.3}$$

in the basis of solutions $\psi_\pm(x,k)$ such that $\psi_\pm(0,k) = 1$, $\psi'_\pm(0,k) = \pm ik$,

$$\cos\varphi = Re(1/T_1), \tag{2.4}$$

where $T_n = s_{22}^{(n)}(E)e^{ikna}$, $R_n = s_{21}^{(n)}(E)$, $\varphi = \varphi(E)$ is the Bloch phase from (1.15), (1.16), $E = k^2$, $k \in \mathbb{R}_+$.

The formulas (2.1)–(2.4) (taking into account (1.8)) describe relations between the scattering matrix $s_{ij}^{(n)}(k)$, $k \in \mathbb{R}_+$, for (1.1) and spectral data for (1.13) in a very complete way. A proper discussion is given in [SWM]. Some similar results are given also in [RRT]. Conserning more old results in this direction see [R] and references given in [SWM], [RRT]. The discrete spectrum for the scattering problem for (1.1) on the whole line was discussed in [R], [SV]. The paper [R] deals with the particular case when $q_1(\frac{a}{2} + x) = q_1(\frac{a}{2} - x)$ and results of [R] conserning the discrete spectrum imply a lower bound for the distribution function of discrete spectrum $F_{sc}^{(n)}(\Sigma)$ for this case. In [SV] the total number of bound states for an n-cell scatterer q_n is given in terms of certain quantities characterizing the single scatterer q_1. However, in [SV] the distribution function

$$F_{sc}^{(n)}(\Sigma) = \ \#\{E_j^{(n)} \in \Sigma\} \tag{2.5}$$

(the number of bound states with energies in an interval $\Sigma \subset]-\infty, 0[$) is not considered for $\Sigma \neq]-\infty, 0[$ and manifestations of the Bloch zone structure for $E_j^{(n)}$ are not discussed.

In the present paper we obtain, in particular, the following result.

Theorem 1. *Under assumptions* (1.2), (1.3), (1.14)*, the following formulas hold:*

$$\left| F_{sc}^{(n)}(]-\infty, E[) - \left[\frac{nap(E)}{\pi}\right] \right| \leq 1 \quad \textit{for } E \in]-\infty, 0]\,, \tag{2.6}$$

$$\left[\frac{nap(E)}{\pi}\right] \leq F_{sc}^{(n)}(]-\infty, E[) \leq \left[\frac{nap(E)}{\pi}\right] + 1 \quad \textit{for } E \in]-\infty, 0] \backslash \bar{\Lambda}^f, \tag{2.7}$$

where $F_{sc}^{(n)}(\Sigma)$ is the distribution function of discrete spectrum for the scattering problem (1.1), *$p(E)$ is the real part of the global quasimomentum and $\bar{\Lambda}^f$ is the closure of the forbidden energy set for the spectral problem* (1.13), $[r]$ *is the integer part of $r \geq 0$.*

The proof of **Theorem 1** is given in Section 4.

Using (2.6) we obtain the following corollary.

Corollary 1. *Under assumptions* (1.2), (1.3), (1.14)*, the following formulas hold:*

$$\begin{aligned} &F_{sc}^{(n)}(\bar{\Lambda}_j^f \cap]-\infty, 0[) \leq 2 \quad \textit{for } j \in J, \\ &F_{sc}^{(n)}(\bar{\Lambda}_1^f \cap]-\infty, 0[) \leq 1, \end{aligned} \tag{2.8}$$

where $\bar{\Lambda}_j^f$ *is the closure of the forbidden zone* Λ_j^f *for* (1.13).

The proof of Corollary 1 is given in Section 4.

Consider now the eigenvalues $E_j^{(n)}$ and the distribution function

$$F^{(n)}(\Sigma) = \#\{E_j^{(n)} \in \Sigma\} \tag{2.9}$$

(the number of eigenvalues in an interval $\Sigma \subset \mathbb{R}$) for the spectral problem (1.1), (1.10).

Theorem 2. *Under assumptions* (1.2), (1.3), (1.11), (1.14)*, the following formulas hold:*

$$\left[\frac{nap(E)}{\pi}\right] - 1 \le F^{(n)}(]-\infty, E[) \le \left[\frac{nap(E)}{\pi}\right] \quad \text{for } E \in \mathbb{R},\ \alpha = 0,\ \ \beta = \pi, \tag{2.10a}$$

$$F^{(n)}(]-\infty, E[) = \left[\frac{nap(E)}{\pi}\right] \quad \text{for } E \in \mathbb{R} \setminus \bar{\Lambda}^f,\ \ \alpha = 0,\ \ \beta = \pi, \tag{2.10b}$$

$$\left|F^{(n)}(]-\infty, E[) - \left[\frac{nap(E)}{\pi}\right]\right| \le 1 \quad \text{for } E \in \mathbb{R},\ \ \alpha < \beta, \tag{2.11a}$$

$$\left[\frac{nap(E)}{\pi}\right] \le F^{(n)}(]-\infty, E[) \le \left[\frac{nap(E)}{\pi}\right] + 1 \quad \text{for } E \in \mathbb{R} \setminus \bar{\Lambda}^f,\ \alpha < \beta, \tag{2.11b}$$

$$\left|F^{(n)}(]-\infty, E[) - \left[\frac{nap(E)}{\pi}\right] - 1\right| \le 1 \quad \text{for } E \in \mathbb{R},\ \ \beta < \alpha, \tag{2.12a}$$

$$\left[\frac{nap(E)}{\pi}\right] + 1 \le F^{(n)}(]-\infty, E[) \le \left[\frac{nap(E)}{\pi}\right] + 2 \quad \text{for } E \in \mathbb{R} \setminus \bar{\Lambda}^f,\ \beta < \alpha, \tag{2.12b}$$

$$\left[\frac{nap(E)}{\pi}\right] \le F^{(n)}(]-\infty, E[) \le \left[\frac{nap(E)}{\pi}\right] + 1 \quad \text{for } E \in \mathbb{R},\ \alpha = \beta, \tag{2.13a}$$

$$F^{(n)}(]-\infty, E[) = \left[\frac{nap(E)}{\pi}\right] + 1 \quad \text{for } E \in \mathbb{R} \setminus \bar{\Lambda}^f,\ \ \alpha = \beta, \tag{2.13b}$$

where $F^{(n)}(\Sigma)$ *is the distribution function for the spectral problem* (1.1), (1.10), $p(E)$ *is the real part of the global quasimomentum and* $\bar{\Lambda}^f$ *is the closure of the forbidden energy set for the spectral problem* (1.13), $[r]$ *is the integer part of* $r \ge 0$.

The proof of **Theorem 2** is given in Section 4.

Using (2.10)–(2.13) we obtain the following corollary.

Corollary 2. *Under assumptions* (1.2), (1.3), (1.11), (1.14), *the following formulas hold:*

$$F^{(n)}(\bar{\Lambda}_1^f) = 0 \quad for\ \alpha = 0,\ \ \beta = \pi, \tag{2.14a}$$

$$F^{(n)}(\bar{\Lambda}_j^f) = 1 \quad for\ j \in J \setminus 1,\ \alpha = 0,\ \ \beta = \pi, \tag{2.14b}$$

$$F^{(n)}(\bar{\Lambda}_j^f) \le 2 \quad for\ j \in J,\ \alpha < \beta, \tag{2.15a}$$

$$F^{(n)}(\bar{\Lambda}_1^f) \le 1 \quad for\ \alpha < \beta, \tag{2.15b}$$

$$F^{(n)}(\bar{\Lambda}_j^f) \le 2 \quad for\ j \in J,\ \beta < \alpha, \tag{2.16a}$$

$$F^{(n)}(\bar{\Lambda}_j^f) = 1 \quad for\ j \in J,\ \alpha = \beta, \tag{2.16b}$$

where $\bar{\Lambda}_j^f$ *is the closure of the forbidden zone* Λ_j *for* (1.13).

The proof of the **Corollary 2** is given in Section 4.

Consider now the eigenvalues $E_j^{(n)}$ for (1.1) with (1.12a), the eigenvalues $\tilde{E}_j^{(n)}$ for (1.1) with (1.12b), and the related distribution functions

$$F^{(n)}(\Omega) = \sum_{E_j^{(n)} \in \Omega} m(E_j^{(n)}), \qquad \tilde{F}^{(n)}(\Omega) = \sum_{\tilde{E}_j^{(n)} \in \Omega} m(\tilde{E}_j^{(n)}), \tag{2.17}$$

where Ω is a subset of $\mathbb{R}$, $m(E_j^{(n)})$, $m(\tilde{E}_j^{(n)}) \in \{1, 2\}$ are the multiplicities of $E_j^{(n)}$, $\tilde{E}_j^{(n)}$.

Under assumptions (1.2), (1.3), (1.14), the following statements are valid:

$$\begin{aligned} &\textit{a number } E \textit{ is a simple eigenvalue for (1.1) with (1.12a) iff} \\ &(2\pi)^{-1} nap(E) \in \mathbb{N} \cup 0, \quad E \in \bar{\Lambda}^f \setminus \Lambda^f, \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\text{a number E is a double eigenvalue for (1.1) with (1.12a) iff} \\ &(2\pi)^{-1} nap(E) \in \mathbb{N} \cup 0, \quad E \in \mathbb{R} \setminus \bar{\Lambda}^f, \end{aligned} \tag{2.19}$$

$$\begin{aligned} &\text{a number E is a simple eigenvalue for (1.1) with (1.12b) iff} \\ &(2\pi)^{-1} (nap(E) - \pi) \in \mathbb{N} \cup 0, \quad E \in \bar{\Lambda}^f \setminus \Lambda^f, \end{aligned} \tag{2.20}$$

a number E is a double eigenvalue for (1.1) with (1.12b) iff

$$(2\pi)^{-1}(nap(E)-\pi)\in\mathbb{N}\cup 0,\quad E\in\mathbb{R}\setminus\bar{\Lambda}^f, \tag{2.21}$$

if $F^{(n)}(\Omega)$ is the distribution function for (1.1) with $(1.12a)$, then

$$\begin{aligned}&[(2\pi)^{-1}nap(E)]\le F^{(n)}(]-\infty,E])\le[(2\pi)^{-1}nap(E)]+1,\\ &F^{(n)}(\Lambda^f)=0,\end{aligned} \tag{2.22}$$

if $\tilde{F}^{(n)}(\Omega)$ is the distribution function for (1.1) with $(1.12b)$, then

$$\begin{aligned}&[(2\pi)^{-1}nap(E)]\le \tilde{F}^{(n)}(]-\infty,E])\le[(2\pi)^{-1}nap(E)]+1,\\ &\tilde{F}^{(n)}(\Lambda^f)=0,\end{aligned} \tag{2.23}$$

where $p(E)$ is the real part of the global quasimomentum and $\bar{\Lambda}^f$ is the closure of the forbidden energy set Λ^f for (1.13), $[r]$ is the integer part of $r\ge 0$.

Using known properties of $p(E)$ and $E_j = E_j^{(1)}$, $\tilde{E}_j = \tilde{E}_j^{(1)}$ (see, for example, [RSS], §17 and [CL], Chapter 8), we obtain these statements, first, for $n=1$. Then we reduce the general case to the case $n=1$ considering q from (1.14) as a potential with period na. One can obtain also these statements using well-known results presented in Chapter 21 of [T] and the definitions of $p(E)$ and $F^{(n)}(\Omega)$, $\tilde{F}^{(n)}(\Omega)$.

Consider now the points of perfect transmission (transmission resonances) for the scattering problem (1.1), i.e. the points $\lambda_j^{(n)}\in\mathbb{R}_+$ such that $|s_{ii}^{(n)}(\lambda_j^{(n)})|=1$.

In [SWM] it is shown (using (2.1)) that, for $E\in\mathbb{R}_+$,

$$E\quad \text{is a point of perfect transmission, i.e.}\quad |s_{ii}^{(n)}(E)|=1,$$

$$\textbf{either}\quad \textbf{if}\ \ |s_{ii}^{(1)}(E)|=1\quad \textbf{or}\quad \textbf{if}\ \ \sin n\varphi(E)=0,\ \sin\varphi(E)\ne 0, \tag{2.24}$$

where $\varphi(E)$ is defined by (1.16). The same result is given also in [RRT].

In the present paper in connection with transmission resonances we obtain the following result.

Proposition 1. *Under assumptions* (1.2), (1.3), (1.14), *the following statements are valid:*

$$\begin{aligned}&\textit{if } E\in\mathbb{R}_+ \textit{ is a double eigenvalue for } (1.1) \textit{ with } (1.12a) \textit{ or with } (1.12b),\\ &\textit{then } |s_{ii}^{(n)}(E)|=1,\end{aligned} \tag{2.25}$$

$$\begin{aligned}&\textit{if } \sin n\varphi(E)=0,\ \sin\varphi(E)\neq 0,\\ &\textit{then } E \textit{ is a double eigenvalue for } (1.1) \textit{ with } (1.12a) \textit{ or with } (1.12b),\end{aligned} \tag{2.26}$$

$$\begin{aligned}&\textit{if } |s_{ii}^{(n)}(E)|=1,\ \ E\in\mathbb{R}_+,\\ &\textit{then } E\in\Lambda^a,\end{aligned} \tag{2.27}$$

where $s_{ii}^{(n)}(E)$ *is the transmission coefficient for* (1.1), $\varphi(E)$ *is the Bloch phase and* Λ^a *is the allowed energy set for* (1.13).

To prove the statement (2.25) we calculate $s_{ii}^{(n)}(E)$ using that in this case each solution of (1.1) on $[0, na]$ satisfies (1.12a) or (1.12b). The statement (2.26) follows from (2.19), (2.21) and properties of $\varphi(E)$. The statement (2.27) follows for example, from (2.24), (2.4) and properties of $\varphi(E)$.

For $q_1(x)\not\equiv 0$ we consider also the distribution function of transmissions resonances

$$\Phi_{sc}^{(n)}(\Sigma) = \ \#\{\lambda_j^{(n)}\in\Sigma\} \ \text{(the number of transmission resonances in } \Sigma\subset\mathbb{R}_+). \tag{2.28}$$

Under assumptions (1.2), (1.3), (1.14), as a corollary of **Theorems 1, 2** the statements (2.18)–(2.24) and **Proposition 1** we obtain the following statements:

if $F_{sc}^{(n)}(\Sigma)$ *is the distribution function of discrete spectrum for the scattering problem* (1.1), *then, for* $E\geq 0$,

$$\lim_{n\to\infty}(na)^{-1}F_{sc}^{(n)}(]-\infty,E[)=\pi^{-1}p(E),$$

$$(na)^{-1}F_{sc}^{(n)}(]-\infty,E[)-(na)^{-1}\leq\pi^{-1}p(E)\leq(na)^{-1}F_{sc}^{(n)}(]-\infty,E[)+2(na)^{-1}, \tag{2.29}$$

if $\Phi_{sc}^{(n)}(\Sigma)$ *is the distribution function of transmission resonances for the scattering problem* (1.1), $q_1(x) \not\equiv 0$, *then, for* $E \geq 0$,

$$\lim_{n\to\infty}(na)^{-1}\Phi_{sc}^{(n)}(]0,E]) = \pi^{-1}(p(E)-p(0)),$$
$$(na)^{-1}\Phi_{sc}^{(n)}(]0,E]) - \pi^{-1}(p(E)-p(0)) = O(n^{-1}),\ \text{as } n\to\infty, \tag{2.30}$$

if $F^{(n)}(\Sigma)$ *is the distribution function for* (1.1), *with* (1.10), *then, for* $E\in\mathbb{R}$,

$$\lim_{n\to\infty}(na)^{-1}F^{(n)}(]-\infty,E]) = \pi^{-1}p(E), \tag{2.31a}$$

$$\begin{gathered}(na)^{-1}F^{(n)}(]-\infty,E]) - (na)^{-1} \leq \pi^{-1}p(E) \leq \\ \leq (na)^{-1}F^{(n)}(]-\infty,E]) + 2(na)^{-1}, \\ \text{for } 0\leq\alpha\leq\beta\leq\pi,\ 0<\beta,\ \alpha<\pi,\end{gathered} \tag{2.31b}$$

$$\begin{gathered}(na)^{-1}F^{(n)}(]-\infty,E]) \leq \pi^{-1}p(E) \leq \\ \leq (na)^{-1}F^{(n)}(]-\infty,E]) + 3(na)^{-1}, \\ \text{for } 0<\beta<\alpha<\pi,\end{gathered} \tag{2.31c}$$

if $F^{(n)}(\Sigma)$ *is the distribution function for* (1.1), *with* (1.12a) *or* (1.12b) *then, for* $E\in\mathbb{R}$,

$$\lim_{n\to\infty}(na)^{-1}F^{(n)}(]-\infty,E]) = \pi^{-1}p(E), \tag{2.32a}$$

$$\begin{gathered}(na)^{-1}F^{(n)}(]-\infty,E]) - (na)^{-1} \leq \pi^{-1}p(E) \leq \\ \leq (na)^{-1}F^{(n)}(]-\infty,E]) + (na)^{-1},\end{gathered} \tag{2.32b}$$

where $p(E)$ *is the real part of the global quasimomentum for* (1.13).

The formulas (2.31a), (2.32a) for the case of smooth potential are a particular case of results of [Sh] about density of states of multidimensional selfadjoint elliptic operators with almost periodic coefficients.

Consider now the one-dimensional Schrödinger equation

$$-\psi'' + q(x)\psi = E\psi,\quad x\in\mathbb{R}, \tag{2.33}$$

with a potential consisting of n not necessarily identical cells. More precisely, we suppose that

$$\mathbb{R} = \bigcup_{j=1}^{n} I_j,\quad n\in\mathbb{N}, \tag{2.34a}$$

$$I_1 =]-\infty, x_1], \ I_j = [x_{j-1}, x_j] \ \text{ for } \ 1 < j < n, \ I_n = [x_{n-1}, -\infty[,$$
$$-\infty < x_{j-1} < x_j < +\infty \ \text{ for } \ 1 < j < n, \tag{2.34b}$$

$$q(x) = \sum_{j=1}^{n} q_j(x), \tag{2.34c}$$

$$q_j \in L^1(\mathbb{R}), \quad q_j = \bar{q}_j, \quad supp\, q_j \subseteq I_j \ \text{ for } \ 1 \le j \le n,$$
$$\textstyle\int_{\mathbb{R}} (1+|x|)|q_1(x)|dx < \infty, \quad \int_{\mathbb{R}} (1+|x|)|q_n(x)|dx < \infty. \tag{2.34d}$$

In the present paper we obtain, in particular, the following result.

Theorem 3. *Under assumptions (2.34), the following estimate holds:*

$$|F(]-\infty, E[) - \sum_{j=1}^{n} F_j(]-\infty, E[)| \le n - 1 \ \text{ for } \ E \le 0, \tag{2.35}$$

where $F(]-\infty, E[)$ $(F_j(]-\infty, E[)$, *respectively) denotes the distribution function of discrete spectrum for the scattering problem for (2.33) (for the one-dimensional Schrödinger equation with the potential* q_j, *respectively) on the whole line.*

In addition, for $E = 0$ there is the following estimate

$$1 - n + \sum_{j=1}^{n} F_j(]-\infty, 0[) \le F(]-\infty, 0[) \le \sum_{j=1}^{n} F_j(]-\infty, 0[) \tag{2.36}$$

given earlier in [AKM2] as a development of results of [K] and [SV]. The estimate (2.36) is more precise than (2.35) for $E = 0$. However, for fixed $E < 0$ and, at least, for $n = 2$ the estimate (2.35) is the best possible, in general.

The proof of Theorem 3 is given in Section 4.

3. Auxiliary results.

To prove **Theorems 1, 2, 3** we use auxiliary results given below separated into five parts.

I. We consider the Schrödinger equation

$$-\Psi'' + v(x)\Psi = E\Psi, \ \ x \in \mathbb{R}, \tag{3.1}$$

where

$$v=\bar{v},\quad v\in L^1_{loc}(\mathbb{R}),\quad E\in\mathbb{R}. \tag{3.2}$$

Under assumptions (3.2), the following formulas hold:

$$\Big|\arg(\Psi(x)+i\Psi'(x))\Big|_0^y-\arg(\varphi(x)+i\varphi'(x))\Big|_0^y\Big|<\pi, \tag{3.3}$$

$$0<\frac{\pi}{2}-\arctan\frac{\varphi'(0)}{\varphi(0)}-\arg(\varphi(x)+i\varphi'(x))\Big|_0^y-\pi N(\varphi,]0,y[)\le\pi\quad\text{if } \varphi(0)\neq 0,$$

$$0<-\arg(\varphi(x)+i\varphi'(x))\Big|_0^y-\pi N(\varphi,]0,y[)\le\pi\quad\text{if } \varphi(0)=0, \tag{3.4}$$

for any non-zero real-valued solutions Ψ and φ of (3.1), where $\arg f(x)$ denotes an arbitrary continuously dependent on x branch of the argument of $f(x)$, $\arctan r\in]-\frac{\pi}{2},\ \frac{\pi}{2}[$, for any $r\in\mathbb{R}$, $N(\varphi,]0,y[)$ denotes the number of zeroes of $\varphi(x)$ in $]0,y[,\ y>0$.

For the case of bounded potential these results were used, actually, in Chapter 8 of [CL] and in Section 4 of [JM].

II. Under assumptions (1.2), (1.3), (1.14), the following formulas hold:

$$\Big|\arg(\varphi(x,E)+i\varphi'(x,E))\Big|_0^{na}+nap(E)\Big|<\pi,\quad\text{for}\ \ E\in\bar{\Lambda}^f, \tag{3.5a}$$

$$0\le-\pi^{-1}\arg(\varphi(x,E)+i\varphi'(x,E))\Big|_0^{na}-[\pi^{-1}nap(E)]<1,\quad\text{for}\ \ E\in\mathbb{R}\setminus\bar{\Lambda}^f, \tag{3.5b}$$

for any non-zero real-valued solutions φ of (1.1), where one takes an arbitrary continuously dependent on x branch of the argument, $p(E)$ is the real part of the global quasimomentum and $\bar{\Lambda}^f$ is the closure of the forbidden energy set for (1.13), $[r]$ is the integer part of $r\ge 0$.

The estimate (3.5a) follows from (3.3) and the formula

$$-arg\,(\psi(x,E)+i\psi'(x,E))|_0^{na}=nap(E)$$

for $E\in\bar{\Lambda}^f$ and any non-zero Bloch solution $\psi(x,E)$ of (1.13). We obtain (3.5b) using

(1) the left-hand side of inequalities (3.4),

(2) Theorem 1.2 of Chapter 8 of [CL],

(3) the representation of the monodromy operator $M(E)$ for $E \in \mathbb{R} \backslash \bar{\Lambda}^f$ as the rotation matrix on the angle $ap(E)$ clockwise for an appropriate basis in the space of solutions (identified with the space of the Cauchy data at $x = 0$) to (1.13) at fixed E ,

(4) the fact that the integer part $[-\pi^{-1} arg\,(\chi_1(\varphi(s), \varphi'(s)) + i\chi_2(\varphi(s), \varphi'(s)))|_0^y]$ (where $(\chi_1(\varphi(s), \varphi'(s)), \chi_2(\varphi(s), \varphi'(s)))$ are the coordinates of the Cauchy data
$(\varphi(s), \varphi'(s))$ at $x = s$ of a non-zero real-valued solution φ of (1.13) at fixed E with respect to a fixed (independent of s) basis (in the space of the Cauchy data at $x = s$) for which the change of variables $(\varphi, \varphi') \to (\chi_1, \chi_2)$ has a positive determinant) is independent of the basis.

III. Under assumptions (1.5), the following formula holds:

$$F_{sc}(]-\infty, E[) = N(\varphi_\pm(\bullet, E),]-\infty, \infty[), \quad E \le 0, \tag{3.6}$$

where $F_{sc}(]-\infty, E[)$ is the number of bound states with energies in $]-\infty, E[$ for the equation (1.4), $\varphi_\pm(x, E)$ are solutions of (1.4) such that

$$\begin{aligned} &\varphi_+(x, E) = e^{-\kappa x}(1 + o(1)),\ \kappa = i\sqrt{E} \ge 0, \quad \text{as} \ \ x \to +\infty \\ &\varphi_-(x, E) = e^{\kappa x}(1 + o(1)),\ \kappa = i\sqrt{E} \ge 0, \quad \text{as} \ \ x \to -\infty, \end{aligned}$$

$N(\varphi(\bullet, E),]-\infty, \infty[)$ is the number of zeroes of $\varphi(x, E)$ in $]-\infty, \infty[$ (with respect to x).

If, in addition to (1.5), $v(x) \equiv 0$ for $x < x_1$, then

$$0 \le N(\psi_-(\bullet, E),]-\infty, \infty[) - N(\varphi_-(\bullet, E),]-\infty, \infty[) \le 1, \quad E \le 0, \tag{3.7}$$

where $\psi_-(x, E)$ is the solution of (1.4) such that

$$\psi_-(x, E) = e^{-\kappa x}, \quad \kappa = i\sqrt{E} \ge 0, \quad \text{for} \ \ x < x_1.$$

One can obtain (3.6) generalizing the proof of the Theorem 2.1 of Chapter 8 of [CL] and using properties of $\varphi(x, E)$ given in Lemma 1 of Section 2 of [DT].

The same arguments that prove (3.3), (3.4) prove also (3.7) (taking into account that

$$\frac{\pi}{2} - \arctan \frac{\psi_-'(x_1, E)}{\psi_-(x_1, E)} \ge \frac{\pi}{2} - \arctan \frac{\varphi_-'(x_1, E)}{\varphi_-(x_1, E)},$$

where $\arctan r \in]-\pi/2,\ \pi/2[$ for $r \in \mathbb{R}$).

Remark. For the case when E is a bound state energy and, as a corollary, $\varphi_\pm(x,E)$ is a bound state, the formula (3.6) was mentioned, for example, in §1 of Chapter 1 of [NMPZ]. Completing the present paper we have found that the statement of the formula (3.6) in the general case was given in Proposition 10.3 of [AKM1].

IV. Let

$$\varphi(x,E) = ae^{\kappa x} + be^{-\kappa x} \quad \text{for } x \geq y, \tag{3.8}$$

where $a,b \in \mathbb{R},\ a^2+b^2 \neq 0,\ \kappa > 0,\ y > 0$.

Then

$$\varphi(x) \neq 0 \text{ for } x \geq y \text{ if } \varphi(y) > 0,\ \ \kappa\varphi(y) + \varphi'(y) \geq 0; \tag{3.9a}$$

$$\varphi(x) \ \text{ has a single zero for } x \geq y \text{ if } \varphi(y) \geq 0,\ \ \kappa\varphi(y) + \varphi'(y) < 0; \tag{3.9b}$$

$$\varphi(x) \neq 0 \text{ for } x \geq y \text{ if } \varphi(y) < 0,\ \ \kappa\varphi(y) + \varphi'(y) \leq 0; \tag{3.9c}$$

$$\varphi(x) \ \text{ has a single zero for } x \geq y \text{ if } \varphi(y) \leq 0,\ \ \kappa\varphi(y) + \varphi'(y) > 0; \tag{3.9d}$$

Let

$$\varphi(x) = a + bx \ \text{ for } x \geq y, \tag{3.10}$$

where $a,b \in \mathbb{R},\ a^2+b^2 \neq 0,\ y > 0$.

Then

$$\varphi(x) \neq 0 \text{ for } x \geq y \text{ if } \varphi(y) > 0,\ \ \varphi'(y) \geq 0; \tag{3.11a}$$

$$\varphi(x) \ \text{ has a single zero for } x \geq y \text{ if } \varphi(y) \geq 0,\ \ \varphi'(y) < 0; \tag{3.11b}$$

$$\varphi(x) \neq 0 \text{ for } x \geq y \text{ if } \varphi(y) < 0,\ \ \varphi'(y) \leq 0; \tag{3.11c}$$

$$\varphi(x) \ \text{ has a single zero for } x \geq y \text{ if } \varphi(y) \leq 0,\ \ \varphi'(y) > 0. \tag{3.11d}$$

V. We consider the Schrödinger equation

$$-\Psi'' + v(x)\Psi = E\Psi,\ \ x \in\ [0,y], \tag{3.12}$$

where

$$v \in L^1([0,y]), \quad v = \bar{v},\ \ y > 0, \tag{3.13}$$

with boundary conditions

$$\begin{aligned} &\Psi(0)\cos\alpha - \Psi'(0)\sin\alpha = 0, \\ &\Psi(y)\cos\beta - \Psi'(y)\sin\beta = 0, \quad \alpha\in\mathbb{R}, \quad \beta\in\mathbb{R}. \end{aligned} \tag{3.14}$$

Without loss of generality we may assume

$$0 \le \alpha < \pi, \quad 0 < \beta \le \pi. \tag{3.15}$$

Consider the eigenvalues E_j and the distribution function

$$F(\Sigma) = \#\{E_j \in \Sigma\} \quad \text{(the number of eigenvalues in an interval } \Sigma\subset\mathbb{R}) \tag{3.16}$$

for the spectral problem (3.12), (3.14).

Consider the solution $\varphi(x,E)$ of (3.12) such that

$$\varphi(0,E) = \sin\alpha, \quad \varphi'(0,E) = \cos\alpha. \tag{3.17}$$

Under assumptions (3.13), (3.15), the following formulas hold:

$$F(]-\infty,E]) = [-\pi^{-1}\mathrm{arg}(\varphi+i\varphi')\Big|_0^y] \quad \text{for } \alpha=0,\ \beta=\pi, \tag{3.18}$$

$$[-\pi^{-1}\mathrm{arg}(\varphi+i\varphi')\Big|_0^y] \le F(]-\infty,E]) \le [-\pi^{-1}\mathrm{arg}(\varphi+i\varphi')\Big|_0^y]+1 \quad \text{for } \alpha<\beta, \tag{3.19}$$

$$[-\pi^{-1}\mathrm{arg}(\varphi+i\varphi')\Big|_0^y]+1 \le F(]-\infty,E]) \le [-\pi^{-1}\mathrm{arg}(\varphi+i\varphi')\Big|_0^y]+2 \text{ for } \alpha<\beta, \tag{3.19}$$

$$F(]-\infty,E]) = [-\pi^{-1}\mathrm{arg}(\varphi+i\varphi')\Big|_0^y]+1 \quad \text{for } \alpha=\beta, \tag{3.21}$$

where $[r]$ is defined by (4.6).

For the case of bounded potential one can obtain these results using the proof of Theorem 2.1 of Chapter 8 of [CL].

4. Proofs of Theorems 1, 2, 3 and Corollaries 1, 2.

Proof of Theorem 1. Consider the solution $\varphi(x,E)$ of (1.1) such that

$$\varphi(x,E) = e^{\kappa x}, \quad \kappa = i\sqrt{E} \ge 0, \quad \text{for } x\le 0. \tag{4.1}$$

Note that

$$\varphi(0,E)=1, \quad \varphi'(0,E)=\kappa. \tag{4.2}$$

Due to (3.4), (4.2) the following formulas hold:

$$\arg(\varphi+i\varphi')\Big|_0^y < \frac{\pi}{2} - \arctan\kappa, \tag{4.3}$$

$$\begin{aligned} &N(\varphi,]0,y[) = [-\pi^{-1}\arg(\varphi+i\varphi')\Big|_0^y] \\ &\text{if} \quad -\tfrac{\pi}{2} \le \arctan\tfrac{\varphi'(y)}{\varphi(y)} \le \arctan\kappa, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &N(\varphi,]0,y[) = [-\pi^{-1}\arg(\varphi+i\varphi')\Big|_0^y] + 1 \\ &\text{if} \quad \arctan\kappa < \arctan\tfrac{\varphi'(y)}{\varphi(y)} < \tfrac{\pi}{2}, \end{aligned} \tag{4.5}$$

where $y>0$,

$$\begin{aligned} &[r] \quad \text{is the integer part of } \ r \ \text{ for } \ r \ge 0, \\ &[r] = -1 \qquad \text{for} \qquad -1 < r < 0, \end{aligned} \tag{4.6}$$

$\arctan(\varphi'(y)/\varphi(y)) = -\pi/2$ means that $\varphi(y)=0$. Due to (1.20), (3.5) the following formulas hold:

$$[\pi^{-1}nap(E)] - 1 \le [-\pi^{-1}\arg(\varphi+i\varphi')\Big|_0^{na}] \le [\pi^{-1}nap(E)] \quad \text{for } E \in \bar\Lambda^f, \tag{4.7}$$

$$[-\pi^{-1}\arg(\varphi+i\varphi')\Big|_0^{na}] = [\pi^{-1}nap(E)] \quad \text{for } E \in \mathbb{R}\setminus\Lambda^f, \tag{4.8}$$

where $[r]$ is defined by (4.6) (we recall that $p(E)\ge 0$ for $E\in\mathbb{R}$).

Due to (4.4), (4.5), (4.7), (4.8) the following formulas hold:

$$[\pi^{-1}nap(E)] - 1 \le N(\varphi,]0,na[) \le [\pi^{-1}nap(E)] + 1 \quad \text{for } E \in \bar\Lambda^f, \tag{4.9}$$

$$[\pi^{-1}nap(E)] \le N(\varphi,]0,na[) \le [\pi^{-1}nap(E)] + 1 \quad \text{for } E \in \mathbb{R}\setminus\bar\Lambda^f, \tag{4.10}$$

and, in addition,

$$\begin{aligned} &\text{if } N(\varphi,]0,na[) = [\pi^{-1}nap(E)] + 1, \\ &\text{then } \ \arctan\kappa < \arctan\tfrac{\varphi'(na)}{\varphi(na)} < \tfrac{\pi}{2}, \end{aligned} \tag{4.11}$$

The function $\varphi(x,E)$ is of the form (4.1) for $x \le 0$, of the form (3.8) for $x \ge na, E < 0$, and of the form (3.10) for $x \ge na, E = 0$. Thus, the function $\varphi(x,E)$ has no zeroes for $x \le 0$ and has at most one zero for $x \ge na$. Thus,

$$N(\varphi,]-\infty, na[) = N(\varphi,]0, na[), \tag{4.12a}$$

$$0 \le N(\varphi,]-\infty, \infty[) - N(\varphi,]0, na[) \le 1. \tag{4.12b}$$

From (4.11), (3.9), (3.11), (4.12) it follows that

$$\begin{aligned} &\text{if} \quad N(\varphi,]0, na[) = [\pi^{-1} nap(E)] + 1, \\ &\text{then} \quad N(\varphi,]-\infty, \infty[) = N(\varphi,]0, na[). \end{aligned} \tag{4.13}$$

The formulas (2.6), (2.7) follow from (3.6), (4.9), (4.10), (4.12b), (4.13).

Proof of the Corollary 1. Consider the energies $z_i,\ i = -1, 0, \dots, 2(\#J-1)$, such that

$$z_{-1} = -\infty, \quad \Lambda_j^f =]z_{2j-3}, z_{2j-2}[, \quad j \in J, \tag{4.14}$$

where $\#J$ is the number of forbidden zones. Due to properties of $p(E)$, for any $j \in J$ and $n \in \mathbb{N}$ there is $\delta^{(n)} > 0$ ($\delta^{(n)}$ depends also on $p(E)$ and a) such that

$$\left[\frac{nap(E)}{\pi}\right] = nl_j, \quad l_j \in \mathbb{N} \cup 0, \quad \text{for } E \in \bar{\Lambda}_j^f \cup [z_{2j-3}, z_{2j-3} + \delta^{(n)}[. \tag{4.15}$$

Due to (2.6), (4.15)

$$\begin{aligned} &F_{sc}^{(n)}(]-\infty, E[) \in \{nl_j - 1,\ nl_j,\ nl_j + 1\}, \quad l_j \ge 1, \\ &F_{sc}^{(n)}(]-\infty, E[) \in \{nl_j,\ nl_j + 1\}, \quad l_j = 0, \\ &\text{for } E \in (\bar{\Lambda}_j^f \cup [z_{2j-3}, z_{2j-3} + \delta^{(n)}[\,) \cap]-\infty, 0]. \end{aligned} \tag{4.16}$$

The formula (2.8) follows from (1.20), (4.16) and the fact that $E_j^{(n)} < 0$.

Proof of Theorem 2. Consider the solution $\varphi(x,E)$ of (1.1) such that

$$\varphi(0,E) = \sin\alpha, \quad \varphi'(0,E) = \cos\alpha. \tag{4.17}$$

Due to (1.20), (3.5) the following formulas hold:

$$[\pi^{-1} nap(E)] - 1 \le [-\pi^{-1} \arg(\varphi + i\varphi')\Big|_0^{na}] \le [\pi^{-1} nap(E)] \quad \text{for } E \in \bar{\Lambda}^f, \tag{4.18}$$

$$[-\pi^{-1}\arg(\varphi+i\varphi')\Big|_0^{na}] = [\pi^{-1}nap(E)] \quad \text{for } E\in\mathbb{R}\setminus\bar\Lambda^f, \tag{4.19}$$

where $[r]$ is defined by (4.6). The formulas (2.10)–(2.13) follow from (3.19)–(3.21), (4.18), (4.19).

Proof of Corollary 2. Due to properties of $p(E)$, for any $j\in J\setminus 1$ and $n\in\mathbb{N}$ there is $\varepsilon^{(n)}>0$ ($\varepsilon^{(n)}$ depends also on $p(E)$ and a) such that

$$\left[\frac{nap(z_{2j-2})}{\pi}\right] - \left[\frac{nap(z_{2j-3}-\varepsilon)}{\pi}\right] = 1 \tag{4.20}$$

for $0<\varepsilon\le\varepsilon^{(n)}$, where z_i are the same as in the proof of **Corollary 1**.

The formula (2.14b) follows from (2.10), (4.20). The formula (2.14a) follows from (2.10a) and (1.20) with $j=1$.

Due to (2.11), (4.20), (1.20), for $\alpha<\beta$,

$$\begin{aligned}
&F^{(n)}(]-\infty,E[) \in \{nl_j-1,\ nl_j,\ nl_j+1\}, \quad \text{for } j\in J\setminus 1,\\
&E\in]z_{2j-3}+\varepsilon^{(n)}[\cup\bar\Lambda^f_j,\\
&F^{(n)}(]-\infty,E[) \in \{nl_j,\ nl_j+1\}, \quad \text{for } j=1,\ E\in\bar\Lambda^f_1.
\end{aligned} \tag{4.21}$$

The formula (2.15) follows from (4.21).

The deduction of others formulas of **Corollary 2** is similar.

Proof of Theorem 3. Suppose, first, that $n=2$. Consider the solution $\varphi_+(x,E)$ of (2.33) such that

$$\varphi_+(x,E) = e^{-\kappa x}(1+o(1)) \ \text{ as } \ x\to+\infty,$$

where (here and below in this proof) $\kappa = i\sqrt{E}\ge 0$.

Note that

$$\varphi_+(x,E) = \varphi_{+,2}(x,E) \ \text{ for } \ x\ge x_1, \tag{4.22}$$

where (here and below in this proof) $\varphi_{\pm,j}$, $j=1,2$, denotes the solution of (1,4) with $v=q_j$ such that

$$\varphi_{+,j}(x,E) = e^{-\kappa x}(1+o(1)) \ \text{ as } \ x\to+\infty,$$

$$\varphi_{-,j}(x,E) = e^{\kappa x}(1+o(1)) \ \text{ as } \ x\to-\infty.$$

Using (3.6) for $v = q_j$ and (4.22) we obtain that

$$N(\varphi_+(\cdot, E), [x_1, +\infty[) \le F_2(]-\infty, E[), \tag{4.23}$$

$$N(\varphi_{-,1}(\cdot, E),]-\infty, x_1[) \le F_1(]-\infty, E[), \tag{4.24}$$

where (here and below in this proof) $N(\varphi(\cdot, E), I)$ denotes the number of zeros of $\varphi(x, E)$ in an interval I (with respect to x). Using the interlacing property of zeros of solutions to (1.4) (see §1 of Chapter 8 of [CL]) we obtain that

$$N(\varphi_+(\cdot, E),]-\infty, x_1[) \le N(\varphi_{-,1}(\cdot, E),]-\infty, x_1[) + 1. \tag{4.25}$$

From (4.23)-(4.25) it follows that

$$N(\varphi_+(\cdot, E),]-\infty, +\infty[) \le F_1(]-\infty, E[) + F_2(]-\infty, E[) + 1. \tag{4.26}$$

Consider now the solution $\varphi_{x_1}(x, E)$ of (2.33) such that

$$\varphi_{x_1}(x_1, E) = e^{-\kappa x_1}, \quad \varphi'_{x_1}(x_1, E) = -\kappa e^{-\kappa x_1}.$$

Note that

$$\begin{aligned} &\varphi_{x_1}(x, E) = \varphi_{+,1}(x, E) \ \text{ for } \ x \le x_1, \\ &\varphi_{x_1}(x, E) = \psi_{-,2}(x, E) \ \text{ for } \ x \ge x_1, \end{aligned} \tag{4.27}$$

where $\psi_{-,2}(x, E)$ is the solution of (1,4) with $v = q_2$ such that

$$\psi_{-,2}(x, E) = e^{-\kappa x} \ \text{ for } \ x \le x_1.$$

Using (3.6) for $v = q_j$ and (3.7) for $v = q_2$ we obtain that

$$\begin{aligned} &N(\varphi_{+,1}(\cdot, E),]-\infty, x_1[) = F_1(]-\infty, E[), \\ &N(\psi_{-,2}(\cdot, E),]x_1, +\infty[) \ge F_2(]-\infty, E[). \end{aligned} \tag{4.28}$$

From (4.27), (4.28) it follows that

$$N(\varphi_{x_1}(\cdot, E),]-\infty, +\infty[) \ge F_1(]-\infty, E[) + F_2(]-\infty, E[). \tag{4.29}$$

Using the interlacing property of zeros of solutions to (1.4) we obtain that

$$N(\varphi_+(\cdot, E),]-\infty, +\infty[) \ge N(\varphi_{x_1}(\cdot, E),]-\infty, +\infty[) - 1. \tag{4.30}$$

From (4.29), (4.30) it follows that

$$F_1(]-\infty,E[)+F_2(]-\infty,E[)-1\le N(\varphi_+(\cdot,E),]-\infty,+\infty[). \qquad (4.31)$$

From (3.6), (4.26), (4.31) it follows that

$$|F(]-\infty,E[)-\sum_{j=1}^{2}F_j(]-\infty,E[)|\le 1.$$

Thus, (2.35) is proved for $n=2$.

We obtain (2.35) for the general case by induction.

The proof of Theorem 3 is completed.

Remark. The main idea of the proof of Theorem 3 is similar to the main idea of the short proof of (2.36) presented in [AKM2] (with a reference to a referee of [AKM2]).

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